BULK DIFFUSION OF 1D EXCLUSION PROCESS WITH BOND DISORDER.

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Abstract. Given a doubly infinite sequence of positive numbers $\{c_k : k \in \mathbb{Z}\}$ such that $\{c_k^{-1} : k \in \mathbb{Z}\}$ satisfies a LLN with limit $\alpha \in (0,\infty]$, we consider the nearest–neighbor simple exclusion process on $\mathbb{Z}$ where c_k is the probability rate of jumps between k and $k+1$. If $\alpha = \infty$ we require an additional minor technical condition. By extending a method developed in [11] we show that the diffusively rescaled process has hydrodynamic behavior described by the heat equation with diffusion constant $1/\alpha$. In particular, the process has diffusive behavior for $\alpha < \infty$ and subdiffusive behavior for $\alpha = \infty$.

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1. Introduction

We consider a particle system on $\mathbb{Z}$ with site exclusion interaction performing a stochastic dynamics with Markov generator

$$\mathcal{L}f(\eta) = \sum_{k\in\mathbb{Z}} c_k \left(f\left(\eta^{k,k+1}\right) - f(\eta)\right), \tag{1}$$

where f is a cylinder function on the state space $\{0,1\}^{\mathbb{Z}}$ and, given $\eta \in \{0,1\}^{\mathbb{Z}}$, $\eta^{k,k+1}$ is defined as

$$\eta_x^{k,k+1} = \begin{cases} \eta_{k+1} & \text{if } x = k, \\ \eta_k & \text{if } x = k+1, \\ \eta_x & \text{otherwise.} \end{cases}$$

The family $\{c_k\}_{k\in\mathbb{Z}}$ is thought of as the environment of the above exclusion process. We assume that $c_k > 0$ for all $k \in \mathbb{Z}$ and that for a suitable constant $\alpha \in (0,\infty]$

$$\lim_{k\uparrow\infty} \frac{S(\lfloor yk\rfloor) - S(\lfloor xk\rfloor)}{(y-x)k} = \alpha \qquad \forall x < y, \tag{2}$$

where $\lfloor\cdot\rfloor$ denotes the integer part and the function $S : \mathbb{Z} \to \mathbb{R}$ is defined as

$$S(k) = \begin{cases} \sum_{j=0}^{k-1} \frac{1}{c_j} & \text{if } k \geq 1, \\ 0 & \text{if } k = 0, \\ -\sum_{j=1}^{-k} \frac{1}{c_{-j}} & \text{if } k < 0. \end{cases} \tag{3}$$

Trivially (2) implies that

$$\lim_{k\uparrow\infty} \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{c_j} = \alpha\,, \qquad \lim_{k\uparrow\infty} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{c_{-j}} = \alpha. \tag{4}$$

If $\alpha \in (0,\infty)$, then (2) and (4) are equivalent. If $\alpha = \infty$, then (4) does not imply (2) (see Appendix A).

Our main results concern the hydrodynamic behavior of the above exclusion process. In what follows, given a probability measure μ on $\{0,1\}^{\mathbb{Z}}$, we denote by $\mathbb{P}_\mu$ the law of the exclusion process with generator (1) and initial distribution μ.

Theorem 1. *Suppose that $\{c_k\}_{k\in\mathbb{Z}}$ satisfies condition (4) with $\alpha \in (0,\infty)$. Let $\rho_0 : \mathbb{R} \to [0,\infty)$ be a bounded Borel function and let $\{\mu_n\}_{n\geq 0}$ be a family of probability measures on $\{0,1\}^{\mathbb{Z}}$ such that, for all $\varphi \in C_0(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\delta > 0$,*

$$\lim_{n\uparrow\infty} \mu_n \left(\left| \frac{1}{n} \sum_{j\in\mathbb{Z}} \varphi\left(\frac{j}{n}\right) \eta_j - \int_{\mathbb{R}} \varphi(x)\rho_0(x)dx \right| > \delta \right) = 0.$$

Then, for all $t > 0$, $\varphi \in C_c(\mathbb{R})$ and $\delta > 0$,

$$\lim_{n\uparrow\infty} \mathbb{P}_{\mu_n} \left(\left| \frac{1}{n} \sum_{j\in\mathbb{Z}} \varphi\left(\frac{j}{n}\right) \eta_j(n^2 t) - \int_{\mathbb{R}} \varphi(x)\rho(x,t)dx \right| > \delta \right) = 0, \tag{5}$$

where $\rho : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ solves the heat equation

$$\partial_t \rho = (1/\alpha)\partial_x^2 \rho$$

with boundary condition ρ_0 at time $t = 0$.

We remark that the above function ρ can be represented as

$$\rho(x,t) = \int_{\mathbb{R}} p(t, x-y)\rho_0(y)dy, \tag{6}$$

where $p(t,x)$ is the density of a Gaussian variable $\mathcal{N}(0, 2t/\alpha)$.

Note that if $\{c_k\}_{k\in\mathbb{Z}}$ are i.i.d. positive random variables such that $\mathbb{E}(1/c_k) < \infty$, then the above theorem holds for almost all realizations of $\{c_k\}_{k\in\mathbb{Z}}$ with $\alpha = \mathbb{E}(1/c_k)$.

In order to discuss the subdiffusive behavior of the system it is convenient to introduce the following notation: we say that condition (H) is fulfilled if for all $x \in \mathbb{Q}$, $a \neq 0$, $\varepsilon > 0$ there exists a sequence of integer numbers $\{b_n\}_{n\geq 1}$ such that $ab_n \geq 0$ and

$$\lim_{n\uparrow\infty} \frac{b_n + |a|/a}{n} \frac{S(\lfloor an \rfloor + \lfloor xn \rfloor) - S(\lfloor xn \rfloor)}{n} = \infty, \tag{7}$$

$$\overline{\lim_{n\uparrow\infty}} \frac{S(b_n + \lfloor xn \rfloor) - S(\lfloor xn \rfloor)}{S(\lfloor an \rfloor + \lfloor xn \rfloor) - S(\lfloor xn \rfloor)} \leq \varepsilon. \tag{8}$$

Note that condition (H) is satisfied if (2) is true with $\alpha = \infty$ and the following holds for all $x \in \mathbb{Q}$, $a \neq 0$:

$$\lim_{\gamma\downarrow 0} \overline{\lim_{n\uparrow\infty}} \frac{S(\lfloor \gamma an \rfloor + \lfloor xn \rfloor) - S(\lfloor xn \rfloor)}{S(\lfloor an \rfloor + \lfloor xn \rfloor) - S(\lfloor xn \rfloor)} = 0. \tag{9}$$

Moreover, note that condition (H) implies (2) with $\alpha = \infty$.

Theorem 2. *Suppose that $\{c_k\}_{k\in\mathbb{Z}}$ satisfies condition (H). Let $\rho_0 : \mathbb{R} \to [0,\infty)$ be in $L^1_{loc}(\mathbb{R})$ and let $\{\mu_n\}_{n\geq 0}$ be a family of probability measures on $\{0,1\}^{\mathbb{Z}}$ such that, for all $\varphi \in C_c(\mathbb{R})$ and $\delta > 0$,*

$$\lim_{n\uparrow\infty} \mu_n \left(\left| \frac{1}{n} \sum_{j\in\mathbb{Z}} \varphi\left(\frac{j}{n}\right) \eta_j - \int_{\mathbb{R}} \varphi(x)\rho_0(x)dx \right| > \delta \right) = 0. \tag{10}$$

Then, for all $t > 0$, $\varphi \in C_c(\mathbb{R})$ and $\delta > 0$,

$$\lim_{n\uparrow\infty} \mathbb{P}_{\mu_n} \left(\left| \frac{1}{n} \sum_{j\in\mathbb{Z}} \varphi\left(\frac{j}{n}\right) \eta_j(n^2 t) - \int_{\mathbb{R}} \varphi(x)\rho_0(x)dx \right| > \delta \right) = 0. \tag{11}$$

We point out that condition (H) enters only in the proof of Proposition 6, which can be obtained under weaker conditions. For example, as discussed in Remark 1, it is enough that there exists an increasing sequence of positive integers $\{n_k\}_{k\geq 1}$ such that

$$\lim_{k\uparrow\infty} \frac{n_{k+1} - n_k}{n_k} = 0 \tag{12}$$

and such that for all $x \in \mathbb{Q}$, $a \neq 0$, $\varepsilon > 0$ one can define a sequence of integer numbers $\{b_k\}_{k\geq 1}$ such that $ab_k \geq 0$ and

$$\lim_{k\uparrow\infty} \frac{b_k + |a|/a}{n_k} \frac{S(\lfloor an_k\rfloor + \lfloor xn_k\rfloor) - S(\lfloor xn_k\rfloor)}{n_k} = \infty, \tag{13}$$

$$\varlimsup_{k\uparrow\infty} \frac{S(b_k + \lfloor xn_k\rfloor) - S(\lfloor xn_k\rfloor)}{S(\lfloor an_k\rfloor + \lfloor xn_k\rfloor) - S(\lfloor xn_k\rfloor)} \leq \varepsilon. \tag{14}$$

In particular, whenever the above condition is fulfilled the particle system has subdiffusive behavior (note that in this case condition (2) with $\alpha = \infty$ is satisfied). As example of application we prove in Section 6 the following result:

Proposition 1. *Suppose that $\{c_k\}_{k\in\mathbb{Z}}$ are i.i.d. positive random variables in the domain of attraction of a ν–stable law with $0 < \nu < 1$. Then the particle system is subdiffusive.*

The above exclusion process with bond disorder is an example of random barrier model (a small transition rate c_k corresponds to a barrier between sites k and $k+1$) and it has been used by physicists to model transport of charge carriers in one dimensional disordered media (see for example [1], [2]). From a physical viewpoint, (2) is the natural condition in order to observe a diffusive behavior possibly with zero diffusion constant: the diffusively rescaled process can be associated to a 1D resistor network with $\mathbb{Z}/n$ as vertex set such that the bond $[j/n, (j+1)/n]$ has resistance $1/(nc_j)$. Then the total resistance of the filament $(x, y]$ is given by

$$\sum_{j=\lfloor xn\rfloor+1}^{\lfloor yn\rfloor} \frac{1}{nc_j}$$

and due to (2) it converges to $\alpha(y - x)$ as $n \uparrow \infty$. Therefore, assumption (2) means that the linear filament has uniform (macroscopic) resistance per unit length equal to α. In particular, it is natural to have a non trivial diffusive behavior if $\alpha < \infty$ and a null diffusive behavior if $\alpha = \infty$ (condition (H) is a more technical condition, used only in the proof of Proposition 6).

Due to the above observation the conditions required in [11][Theorem 3] appear artificial. There, K. Nagy proves the same result as in Theorem 1 above for almost all realization of a i.i.d. random environment $\{c_k\}_{k\in\mathbb{Z}}$ by requiring that $E(c_k^{-4}) < \infty$ and that $c_k \leq C < \infty$ a.s. The strategy followed by K. Nagy consists in showing that, for what concerns bulk diffusion, one can ignore the site exclusion constraint in the diffusive limit. In these notes we show how to improve this method by using a classing result of C. Stone [12] allowing to represent the random walk on $\mathbb{Z}$ having c_k as probability rate of jumps between $k, k+1$ as a space–time change of a 1D Brownian motion (see also [8]).

We observe that by techniques which are standard for non gradient systems one can prove the hydrodynamic limit for the nearest–neighbor exclusion process on $\mathbb{Z}^d$ with bond disorder, where $c_{x,y}$ is the probability rate for a jump between adjacent sites x, y and $\{c_{x,y} : |x-y| = 1\}$ is a family of i.i.d. random variables such that $0 < C \le c_{x,y} \le C^{-1}$ a.s. The hydrodynamic limit holds for almost any realization of the disorder $\{c_{x,y} : |x-y| = 1\}$ and is independent from the disorder. See for example [5] (here the canonical expectation of the gradient density field is zero, thus simplifying drastically the treatment in [5] and allowing to get easily a proof for any dimension).

We point out that the results of Section 3 are valid in all dimensions, while Stone's method (treated in Section 4) works only in dimension one. In particular, the method described here allows to prove the hydrodynamic limit of the exclusion process with bond disorder in any dimension d when having results on the single random walk similar to the ones described in Section 4. For a more detailful discussion see Appendix B.

The hydrodynamic behaviour of one–dimensional stochastic processes with disorder have been studied in several papers (e.g. [6], [9]). For a discussion on the hydrodynamic limit of lattice gases with site disorder see [5] and references therein.

The paper is structured as follows. In Section 2 we show that the dynamics of the above exclusion process is well defined and recall its graphical representation. In Section 3 we recall and extend the method developed in [11][Section 4]. In Section 4 we study the symmetric random walk on $\mathbb{Z}$ with rates $\{c_k\}_{k\in\mathbb{Z}}$ using Stone's representation. In Section 5 we give the proof of Theorem 1 and Theorem 2, while in Section 6 we give the proof of Proposition 1. Finally in Appendix A we show that condition (2) is not equivalent to condition (4) if $\alpha = \infty$ and in Appendix B we show some extensions of our results to higher dimension.

2. Graphical representation of the exclusion process

By means of the graphical representation of exclusion processes [3] [10], we prove in this section that the dynamics of the exclusion process with generator (1) is well defined since (4) holds with $\alpha \in (0, \infty]$. The graphical representation explained below will be used also in Section 3.

Let $N_k(\cdot)$, $k \in \mathbb{Z}$, be a family of independent Poisson processes defined on some probability space $(\Omega, \mathcal{F}, P)$ such that $E(N_k(t)) = c_k t$. Given $t > 0$ we define $\mathcal{G}_t$ as the random graph with vertex set $\mathbb{Z}$ and edges $\{k, k+1\}$ such that $N_k(t) \ge 1$.

Lemma 1. *For almost all ω, the graph $\mathcal{G}_t(\omega)$ has only finite connected components for all $t > 0$.*

Proof. We claim that $P(N_x(t) \ge 1 \ \forall x \ge k) = 0$ for all $k \in \mathbb{Z}$. In fact, since $1 - z \le e^{-z}$ for all $z \ge 0$,

$$P(N_x(t) \ge 1 \ \forall x \ge k) = \lim_{N\uparrow\infty} \Pi_{x=k}^N \left(1 - e^{-c_x t}\right) \le \lim_{N\uparrow\infty} \exp\left\{ -\sum_{x=k}^N e^{-c_x t} \right\}$$

and the sum in the last member goes to ∞ as $N \uparrow \infty$ since it cannot hold $\lim_{x\uparrow\infty} c_x = \infty$ due to (4).

Similarly one can prove that $P(N_x(t) \ge 1 \ \forall x \le k) = 0$ for all $k \in \mathbb{Z}$. In particular, almost surely for all $t \in \mathbb{N}$ the set $\{x : N_x(t) = 0\}$ is unbounded from below and from above, thus implying that $\mathcal{G}_t$ has only finite connected components for all $t \in \mathbb{N}$. To conclude the proof it is enough to observe that $\mathcal{G}_s \subset \mathcal{G}_t$ for $s \le t$. $\square$

Let $\mathcal{A} \in \mathcal{F}$, with $P(A) = 1$, be a set of configurations ω such that $\mathcal{G}_t(\omega)$ has only finite connected components for all $t > 0$ and $N_k(\cdot)$ has only jumps of value 1 for all $k \in \mathbb{Z}$. Let $\omega \in \mathcal{A}$. Then, given an initial configuration $\eta(0)$, the configuration $\eta(t) = \eta(t)[\omega]$ at time t is defined as follows:

Let $\mathcal{C}$ be any connected component of $\mathcal{G}_t(\omega)$ and let

$$\{s_1 < s_2 < \cdots < s_r\} = \{s \,:\, N_k(s) = N_k(s-) + 1,\ \{k, k+1\} \in \mathcal{C},\ 0 < s \le t\}.$$

Start with $\eta(0)$. At time s_1 switch the values between η_k and η_{k+1} if $N_k(s_1) = N_k(s_1-)+1$ and $\{k, k+1\} \in \mathcal{C}$. Repeat the same operation orderly for times $s_2, s_3, \ldots, s_r$. Then the resulting configuration coincides with $\eta(t)$ on $\mathcal{C}$.

3. Site exclusion constraint

Following the main ideas of [11][Section 4], we prove in this section that the site exclusion constraint becomes negligible when considering the bulk diffusion of the particle system, i.e. from a hydrodynamic viewpoint the system behaves as a family of independent continuous–time random walks on $\mathbb{Z}$ with Markov generator $H : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ defined as

$$(Hf)_k = c_k\,(f_{k+1} - f_k) + c_{k-1}\,(f_{k-1} - f_k)\,. \tag{15}$$

Note that the random walk on $\mathbb{Z}$ with Markov generator H is reversible since the transition rates are bond dependent. In particular, $p(t, j, k) = p(t, k, j)$ where $p(t, x, y)$ denotes the probability that the random walk starting at x is in y at time t.

Since

$$d\eta_k(t) = (\eta_{k+1} - \eta_k)(t-)dN_k(t) + (\eta_{k-1} - \eta_k)(t-)dN_{k-1}(t),$$

we can write

$$d\eta(t) = H\eta(t)dt + dM(t) \tag{16}$$

where

$$dM_k(t) = (\eta_{k+1} - \eta_k)(t-)dA_k(t) + (\eta_{k-1} - \eta_k)(t-)dA_{k-1}(t)$$

and

$$A_x(t) = N_x(t) - c_x t.$$

Note that $M_k(\cdot)$ has trajectories of bounded variation on finite intervals a.s.

Formally, (16) implies that

$$\eta(t) = T(t)\eta(0) + \int_0^t T(t-s)dM(s) \tag{17}$$

where $T(t) = e^{tH}$, i.e.

$$\eta_k(t) = \sum_{j\in\mathbb{Z}} p(t, k, j)\eta_j(0) + \sum_{j\in\mathbb{Z}} \int_0^t p(t-s, k, j)dM_j(s). \tag{18}$$

Due to the graphical construction of the dynamics discussed in Section 2, if $\sum_{x\in\mathbb{Z}} \eta_x(0) < \infty$, then for all but a finite family of indexes j $dM_j(s) = 0$ for all $0 \le s \le t$ and in particular the last series in (18) reduces to a finite sum and is meaningful. In this case, one can check that (18) holds a.s. by direct computation using that

$$\frac{d}{dt}p(t, k, j) = (Hp(t, \cdot, j))_k\,.$$

The following result shows that the site exclusion constraint is negligible in the diffusive limit from a hydrodynamic viewpoint:

Proposition 2. *Given $\delta > 0$, $t > 0$, $\varphi \in C_c(\mathbb{R})$ and given a sequence of probability measures μ_n on $\{0,1\}^{\mathbb{Z}}$,*

$$\lim_{n\uparrow\infty} \mathbb{P}_{\mu_n}\left(\left| \frac{1}{n}\sum_{k\in\mathbb{Z}} \varphi\left(\frac{k}{n}\right)\eta_k(tn^2) - \frac{1}{n}\sum_{k\in\mathbb{Z}}\varphi\left(\frac{k}{n}\right)\sum_{j\in\mathbb{Z}} p(tn^2,k,j)\eta_j(0) \right| > \delta \right) = 0. \tag{19}$$

Proof. Let the support of φ be included in $[-L, L]$ and fix $\varepsilon > 0$. Given $x \in \mathbb{Z}$ and $t > 0$ define $\mathcal{C}_x(t)$ as the connected component of $\mathcal{G}_t$ containing x. Then for each positive integer n we can choose b_n large enough such that $P(\mathcal{A}_n^c) < \epsilon$ where $\mathcal{A}_n$ is the subset of configurations ω satisfying the following conditions:

$$\cup_{x\in[-Ln,Ln]} C_x(tn^2) \subset [-b_n, b_n], \tag{20}$$

$$(2L+1)\|\varphi\|_\infty \sup_{k\in[-Ln,Ln]} \sum_{j\,:\,|j|>b_n} p(tn^2,k,j) \le \delta/2. \tag{21}$$

Given $\eta(0)$ and n, we define $\eta^{(n)}(0) \in \{0,1\}^{\mathbb{Z}}$ as

$$\eta^{(n)}_k(0) = \eta_k(0)\mathbb{I}_{|k|\le b_n}$$

and write $\eta^{(n)}(s)$ for the configuration at time s obtained by the graphical construction when starting from $\eta^{(n)}(0)$ at time 0.

Due to the graphical construction of the dynamics and condition (20), if $\omega \in \mathcal{A}_n$ then

$$\eta_k(tn^2) = \eta^{(n)}_k(tn^2), \qquad \forall k \in [-Ln, Ln],$$

thus implying

$$\frac{1}{n}\sum_{k\in\mathbb{Z}}\varphi\left(\frac{k}{n}\right)\eta_k(tn^2) = \frac{1}{n}\sum_{k\in\mathbb{Z}}\varphi\left(\frac{k}{n}\right)\eta^{(n)}_k(tn^2).$$

Moreover, due to (21), if $\omega \in \mathcal{A}_n$ then

$$\frac{1}{n}\sum_{k\in\mathbb{Z}}\left|\varphi\left(\frac{k}{n}\right)\right|\sum_{j\in\mathbb{Z}} p(tn^2,k,j)|\eta_j(0) - \eta^{(n)}_j(0)| \le \delta/2.$$

Therefore the l.h.s. of (19) with fixed n can be bounded by

$$\mathbb{P}_{\mu_n}\left(|Z_n| > \delta/2\right) + P(\mathcal{A}_n^c) \le 4\mathbb{E}_{\mu_n}\left(Z_n^2\right)/\delta^2 + \varepsilon$$

where

$$Z_n = \frac{1}{n}\sum_{k\in\mathbb{Z}}\varphi\left(\frac{k}{n}\right)\eta^{(n)}_k(tn^2) - \frac{1}{n}\sum_{k\in\mathbb{Z}}\varphi\left(\frac{k}{n}\right)\sum_{j\in\mathbb{Z}} p(tn^2,k,j)\eta^{(n)}_j(0).$$

Since $\sum_{x\in\mathbb{Z}} \eta^{(n)}_x(0) < \infty$, setting

$$dM^{(n)}_k(s) = (\eta^{(n)}_{k+1} - \eta^{(n)}_k)(s-)dA_k(s) + (\eta^{(n)}_{k-1} - \eta^{(n)}_k)(s-)dA_{k-1}(s),$$

(18) implies that

$$Z_n = \frac{1}{n}\sum_{k\in\mathbb{Z}}\varphi\left(\frac{k}{n}\right)\sum_{j\in\mathbb{Z}}\int_0^{tn^2} p(tn^2-s,k,j)dM^{(n)}_j(s).$$

In order to conclude the proof it is enough to apply Lemma 2 to the above estimates. $\square$

Lemma 2. *For each $n \in \mathbb{N}$ let ν_n be a probability measure on $\{0,1\}^{\mathbb{Z}}$ such that $\nu_n(\sum_{k\in\mathbb{Z}} \eta_k < \infty) = 1$. Then*

$$\lim_{n\uparrow\infty} \mathbb{E}_{\nu_n}\left(\left(\frac{1}{n}\sum_{k\in\mathbb{Z}}\varphi\left(\frac{k}{n}\right)\sum_{j\in\mathbb{Z}}\int_0^{tn^2} p(tn^2-s,k,j)dM_j(s)\right)^2\right) = 0.$$

Recall that the above series over j reduces to a finite sum whenever $\sum_{k\in\mathbb{Z}}\eta_k(0) < \infty$, and therefore it is well defined a.s.

Proof. We define f_n as

$$\begin{aligned} f_n = \frac{1}{n}\sum_{k\in\mathbb{Z}}\varphi\left(\frac{k}{n}\right)\sum_{j\in\mathbb{Z}}\int_0^{tn^2} p(tn^2-s,k,j)dM_j(s) = \\ \frac{1}{n}\sum_{k\in\mathbb{Z}}\varphi\left(\frac{k}{n}\right)\sum_{j\in\mathbb{Z}}\int_0^{tn^2}\big(p(tn^2-s,k,j)-p(tn^2-s,k,j+1)\big)(\eta_{j+1}-\eta_j)(s-)dA_j(s). \end{aligned} \tag{22}$$

We remark that due to the graphical representation of the exclusion process, f_n can be thought of as a function on the probability space $\big(\{0,1\}^{\mathbb{Z}}\times\Omega, \mathcal{B}\times\mathcal{F}, \nu_n\otimes P\big)$, where $\mathcal{B}$ denotes the Borel σ–algebra of $\{0,1\}^{\mathbb{Z}}$. Moreover, note that $|f_n| \le c(\varphi)$ due to (18).

In the following arguments n can be thought of as fixed. Due to our assumption on ν_n, given ε with $0<\varepsilon<1$ there exists $\ell_n \in \mathbb{N}$ such that $\nu_n(A^c) \le \varepsilon$ where

$$A = \{\eta \,:\, \eta_x = 0 \text{ if } |x| \ge \ell_n\}.$$

Moreover, one can find $M \in \mathbb{N}$ such that $P(B^c) \le \varepsilon$ where

$$B = \big\{\omega \,:\, \cup_{|x|\le\ell_n} C_x(tn^2)[\omega] \subset (-M,M)\big\}.$$

Then $(\nu_n\otimes P)(A\times B) \ge (1-\varepsilon)^2$. Due to the graphical representation, one gets

$$\mathbb{I}_D f_n = \mathbb{I}_D f_{M,n}$$

where $D = A\times B$ and

$$\begin{aligned} &f_{M,n} = \\ &\frac{1}{n}\sum_{k\in\mathbb{Z}}\varphi\left(\frac{k}{n}\right)\sum_{j\,:\,|j|\le M}\int_0^{tn^2}\big(p(tn^2-s,k,j)-p(tn^2-s,k,j+1)\big)(\eta_{j+1}-\eta_j)(s-)dA_j(s). \end{aligned}$$

In particular,

$$\mathbb{E}_{\nu_n}(f_n^2) \le c(\varphi)^2\,(\nu_n\times P)\,(D^c) + \mathbb{E}_{\nu_n}\left(\mathbb{I}_D f_{M,n}^2\right) \le c(\varphi)^2(2\varepsilon-\varepsilon^2) + \mathbb{E}_{\nu_n}\left(f_{M,n}^2\right).$$

By the same computations as in [11][Lemma 12], one gets

$$\mathbb{E}_{\nu_n}(f_{M,n}^2) \le \frac{1}{2n}\sum_{j\in\mathbb{Z}}\frac{1}{n}\varphi^2\left(\frac{j}{n}\right),$$

implying that $\overline{\lim}_{n\uparrow\infty} E_{\nu_n}\left(f_n^2\right) \le c(\varphi)^2(2\varepsilon-\varepsilon^2)$. Since ε is arbitrary, we get the thesis. $\square$

4. The random walk on $\mathbb{Z}$ with jump rates $\{c_k\}_{k\in\mathbb{Z}}$

Let us recall how one can express a 1D nearest–neighbor random walk as space–time change of a 1D Brownian motion (see [12] for a detailful and more general discussion).

Let B be a Brownian motion with $\mathbb{E}(B^2(t)) = t$, defined on some probability space $(\mathbb{W}, \mathbb{F}, \mathbb{P})$ (note that in [12] $\mathbb{E}(B^2(t)) = 2t$, thus changing the final results of some factor 2). Denote by $L(t, y)$ the local time of B. Then, $\mathbb{P}$–almost surely,

$$\int_a^b L(t, y)dy = \int_0^t \mathbb{I}_{\{a\leq B(s)\leq b\}} ds \qquad \forall t \geq 0,\ \forall a \leq b\,. \tag{23}$$

Let ν be a Radon measure on $\mathbb{R}$ (i.e. ν is a Borel positive measure on $\mathbb{R}$, bounded on bounded intervals). We write $\mathrm{supp}(\nu)$ for the support of ν and we assume that $\mathrm{supp}(\nu)$ is unbounded from below and from above, namely

$$\inf(\mathrm{supp}(\nu)) = -\infty\,, \qquad \sup(\mathrm{supp}(\nu)) = \infty\,.$$

For each $x \in \mathrm{supp}(\nu)$ and $t \geq 0$, set

$$\psi(t|x, \nu) = \int_{\mathbb{R}} L(t, y - x)\nu(dy), \tag{24}$$

$$\psi^{-1}(t|x, \nu) = \sup\{s \geq 0 \,:\, \psi(s|x, \nu) \leq t\}\,. \tag{25}$$

Finally, we set

$$Z(t|x, \nu) = B\left(\psi^{-1}(t|x, \nu)\right) + x. \tag{26}$$

The process $(Z(t|x, \nu),\, t \geq 0)$, defined on $(\mathbb{W}, \mathbb{F}, \mathbb{P})$ has paths in the Skohorod space $D([0, \infty), \mathbb{R})$ endowed of the Skohorod metric d_S. Due to [12][Theorem 1] and [12][Corollary 1], the following holds

Proposition 3. [12] *Let $\{\nu_n\}_{n\geq 0}$, ν be Radon measures on $\mathbb{R}$ with support unbounded from below and from above and let $x_n \in supp(\nu_n)$ be a converging sequence with $\lim_{n\uparrow\infty} x_n = x$. Suppose that:*
i) $\nu_n([a, b]) \to \nu([a, b])$ for all $a < b$ with $\nu(\{a\}) = \nu(\{b\}) = 0$,
ii) if $y_n \in supp(\nu_n)$ is a converging sequence as $n \uparrow \infty$, then $\lim_{n\uparrow\infty} y_n \in supp(\nu)$.
Then

$$\lim_{n\uparrow\infty} d_S\left(Z(\cdot|x_n, \nu_n), Z(\cdot|x, \nu)\right) = 0\,, \qquad \mathbb{P}\ a.s. \tag{27}$$

Let us recall another consequence of the results in [12] (see also [8][Section 2]):

Proposition 4. [12] *Let $\{x_k\}_{k\in\mathbb{Z}}$ satisfy $\lim_{k\downarrow-\infty} x_k = -\infty$, $\lim_{k\uparrow\infty} x_k = \infty$. Fix positive constants $\{w_k\}_{k\in\mathbb{Z}}$ and set $\nu = \sum_{k\in\mathbb{Z}} w_k\delta_{x_k}$. Then $Z(\cdot|x_j, \nu)$ is the continuous–time random walk on $\{x_k\}_{k\in\mathbb{Z}}$ starting in x_j such that after reaching site x_k it remains in x_k for an exponential time with mean*

$$2w_k \frac{(x_{k+1} - x_k)(x_k - x_{k-1})}{x_{k+1} - x_{k-1}}$$

and then it jumps to x_{k-1}, x_{k+1} respectively with probability

$$\frac{x_{k+1} - x_k}{x_{k+1} - x_{k-1}} \quad and \quad \frac{x_k - x_{k-1}}{x_{k+1} - x_{k-1}}\,.$$

4.1. **The case $\alpha \in (0,\infty)$.** Recall the definition of S given in (3) and set $S_n(x) = \frac{S(\lfloor xn \rfloor)}{n}$. Due to condition (4)

$$\lim_{k\uparrow\infty} \frac{S(\pm k)}{k} = \alpha, \tag{28}$$

implying that $\lim_{n\uparrow\infty} S_n(x) = \alpha x$ for all $x \in \mathbb{R}$.

Set

$$\nu_n = \frac{1}{2n}\sum_{k\in\mathbb{Z}} \delta_{S(k)/n}\,, \qquad \nu(dx) = \frac{1}{2\alpha}dx.$$

Due to Proposition 4,

$$X_n(t|x) = \frac{1}{n}S^{-1}\left(nZ(t|S_n(x),\nu_n)\right)\,, \qquad x \in \mathbb{Z}/n\,, \tag{29}$$

is the random walk on $\mathbb{Z}/n$ starting at x with generator $H_n : \mathbb{R}^{\mathbb{Z}/n} \to \mathbb{R}^{\mathbb{Z}/n}$ where

$$H_n f\left(\frac{k}{n}\right) = n^2 c_k\left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right) + n^2 c_{k-1}\left(f\left(\frac{k-1}{n}\right) - f\left(\frac{k}{n}\right)\right). \tag{30}$$

Lemma 3. *For all $a<b$*

$$\lim_{n\uparrow\infty} \nu_n([a,b]) = \nu([a,b]).$$

Proof. Since $\lim_{n\uparrow\infty}\nu_n(\{x\}) = 0$ for all $x \in \mathbb{R}$, it is enough to consider the cases where $a=0$ or $b=0$. We deal with the former (the latter is similar).

Trivially, $\nu_n([0,b]) = (\bar{k}+1)/2n$ where $\bar{k} = \max\{k \geq 0 : S(k) \leq nb\}$. Note that $\bar{k} = \bar{k}(n)$. Due to (28), given $\varepsilon > 0$ there exists $k(\varepsilon)$ such that

$$\frac{k}{n}(\alpha-\varepsilon) \leq \frac{S(k)}{n} \leq \frac{k}{n}(\alpha+\varepsilon) \qquad \forall n \geq 1, \forall k \geq k(\varepsilon). \tag{31}$$

Due to (28) $\lim_{n\uparrow\infty}\bar{k}(n) = \infty$, therefore we can assume the above expression to be true for $k = \bar{k}, \bar{k}+1$. Since $S(\bar{k})/n \leq b$ and $S(\bar{k}+1)/n > b$, (31) implies

$$\frac{b}{2(\alpha+\varepsilon)} - \frac{1}{2n} < \frac{\bar{k}}{2n} \leq \frac{b}{2(\alpha-\varepsilon)}$$

and therefore the thesis. □

Due to the above lemma, Proposition 3 holds for all sequences x_n, $n \geq 1$, such that $x_n \in \mathbb{Z}/n$ and $x = \lim_{n\uparrow\infty} x_n$. Moreover, due to (23), $\psi(t|x,(2\alpha)^{-1}dx) = (2\alpha)^{-1}t$ thus implying

$$Z(\cdot|x,(2\alpha)^{-1}dx) = B(2\alpha t) + x \sim \sqrt{2\alpha}B(t) + x,$$

where $X \sim Y$ means that the random variables X, Y have the same law.

The proof of the hydrodynamic limit will be based on the following technical result:

Proposition 5. *Suppose that $\alpha \in (0,\infty)$. Fix $t>0$. Then for all $x \in \mathbb{R}$,*

$$X_n(t|x_n) \Rightarrow \sqrt{2/\alpha}B(t) + x \qquad \textit{as } n \to \infty, \tag{32}$$

where $\Rightarrow$ denotes weak convergence and $x_n = \lfloor xn \rfloor/n$.

Proof. By Proposition 3, since $S_n(x_n) \to \alpha x$, $\mathbb{P}$–almost surely

$$\lim_{n\uparrow\infty} d_S\left(\,Z(\cdot|S_n(x_n),\nu_n), Z(\cdot|\alpha x,(2\alpha)^{-1}dx)\,\right) = 0. \tag{33}$$

Since $\mathbb{P}$–almost surely $Z\left(\cdot|\alpha x, (2\alpha)^{-1}dx\right)$ is continuous, (33) implies that $\mathbb{P}$–almost surely

$$\lim_{n\uparrow\infty} \sup_{0\leq s\leq T} \left|Z(s|S_n(x_n),\nu_n) - Z(s|\alpha x, (2\alpha)^{-1}dx)\right| = 0\,, \qquad \forall T>0. \tag{34}$$

Fix $a\in\mathbb{R}$. Since S is increasing and due to (29),

$$\mathbb{P}\left(X_n(t|x_n)\leq a\right) = \mathbb{P}\left(Z(t|S_n(x_n),\nu_n)\leq S_n(a)\right).$$

Due to (34) with $T=t$ and since, given $\varepsilon>0$, $\alpha a-\varepsilon\leq S_n(a)\leq \alpha a+\varepsilon$ for n large enough, we obtain that

$$\begin{aligned}\mathbb{P}\left(Z(\cdot|\alpha x,(2\alpha)^{-1}dx)\leq \alpha a-\varepsilon\right) &\leq \varliminf_{n\uparrow\infty}\mathbb{P}\left(X_n(t|x_n)\leq a\right)\\ &\leq \varlimsup_{n\uparrow\infty}\mathbb{P}\left(X_n(t|x_n)\leq a\right)\leq \mathbb{P}\left(Z(\cdot|\alpha x,(2\alpha)^{-1}dx)\leq \alpha a+\varepsilon\right). \end{aligned}\tag{35}$$

Due to arbitrariness of ε,

$$\lim_{n\uparrow\infty}\mathbb{P}\left(X_n(t|x_n)\leq a\right) = \mathbb{P}\left(Z(\cdot|\alpha x,(2\alpha)^{-1}dx)\leq \alpha a\right) = \mathbb{P}(\sqrt{2/\alpha}B(t)+x\leq a).$$

□

4.2. **The case $\alpha=\infty$.** Fix $x\in\mathbb{R}$ and define $S^{(n)}:\mathbb{Z}\to\mathbb{R}$ as

$$S^{(n)}\left(\lfloor nx\rfloor+k\right) = S\left(\lfloor nx\rfloor+k\right) - S\left(\lfloor nx\rfloor\right), \qquad \forall k\in\mathbb{Z}.$$

Define the measure ν_n as

$$\nu_n = \frac{1}{2n}\sum_{k\in\mathbb{Z}}\delta_{S^{(n)}(\lfloor nx\rfloor+k)/n}.$$

Let $x_n=\lfloor nx\rfloor/n$. Since

$$S^{(n)}\left(\lfloor nx\rfloor+k+1\right) - S^{(n)}\left(\lfloor nx\rfloor+k\right) = \frac{1}{c_{\lfloor nx\rfloor+k}},$$

Proposition 4 implies that

$$X_n(t|x_n) = \frac{1}{n}\left(S^{(n)}\right)^{-1}\left(nZ(t|0,\nu_n)\right) \tag{36}$$

is the continuous–time random walk on $\mathbb{Z}/n$ starting at x_n with Markov generator H_n defined in (30).

Proposition 6. *Suppose that $\alpha=\infty$ and that the assumptions of Theorem 2 are satisfied. Fix $t>0$. Then for all $x\in\mathbb{R}$,*

$$X_n(t|x_n)\Rightarrow x \qquad \text{as } n\to\infty, \tag{37}$$

where $\Rightarrow$ denotes weak convergence and $x_n=\lfloor xn\rfloor/n$.

Proof. Given n and $u<v<w$ in $\mathbb{Z}/n$, it is simple to build on a same probability space random walks $X_n'(\cdot|u)$, $X_n'(\cdot|v)$, $X_n'(\cdot|w)$ having respectively the same law of $X_n(\cdot|u)$, $X_n(\cdot|v)$, $X_n(\cdot|w)$ and such that

$$X_n'(s|u)\leq X_n'(s|v)\leq X_n'(s|w) \qquad \forall s\geq 0. \tag{38}$$

To this aim consider a family of independent Poisson processes $\left\{N_k^-(\cdot)\right\}_{k\in\mathbb{Z}}$, $\left\{N_k^+(\cdot)\right\}_{k\in\mathbb{Z}}$ such that $E\left(N_k^-(t)\right) = E\left(N_k^+(t)\right) = c_k n^2 t$ for all $k\in\mathbb{Z}$. The random walk on $\mathbb{Z}/n$ starting in a generic point $x\in\mathbb{Z}/n$ can be realized as follows: arrived at a point k/n at time t the particle waits until time s where

$$s=\inf\left\{u>t\,:\,N_k^-(t-)\neq N_k^-(t) \text{ or } N_k^+(t-)\neq N_k^+(t)\right\}.$$

At time s the particle jumps on the left if $N_k^-(t-) \neq N_k^-(t)$ otherwise it jumps on the right.

Due to such a coupling it is enough to prove the thesis for $x \in \mathbb{Q}$. We first prove that for all $a > 0$

$$\lim_{n\uparrow\infty} \mathbb{P}\left(X_n(t|x_n) > x_n + a\right) = 0. \tag{39}$$

Due to (36), it is enough to prove that

$$\lim_{n\uparrow\infty} \mathbb{P}\left(Z(t|0,\nu_n) > w_n\right) = 0, \tag{40}$$

where

$$w_n = \frac{1}{n} S^{(n)}\left(\lfloor na \rfloor + \lfloor nx \rfloor\right).$$

On $(\mathbb{W}, \mathbb{F}, \mathbb{P})$ define the hitting time

$$\tau_y = \inf\left\{s \geq 0 \,:\, B_s = y\right\}.$$

Then the reflection principle implies

$$\mathbb{P}(\tau_y \leq s) = 2\mathbb{P}(B_s \geq y) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-y/\sqrt{s}}^{y/\sqrt{s}} e^{-\frac{z^2}{2}} dz. \tag{41}$$

Due to definition (26), if $Z(t|0,\nu_n) > w_n$ then $\psi^{-1}\left(t|0,\nu_n\right) > \tau_{w_n}$, which implies that $\psi\left(\tau_{w_n}|0,\nu_n\right) \leq t$. Since $\psi(\cdot|0,\nu_n)$ is not decreasing, for all $\delta > 0$,

$$\mathbb{P}\left(Z(t|0,\nu_n) > w_n\right) \leq \mathbb{P}(\tau_{w_n} < \delta w_n^2) + \mathbb{P}\left(\psi(\delta w_n^2 \,|\, 0, \nu_n) \leq t\right). \tag{42}$$

The first addendum in the r.h.s. can be treated by means of (41):

$$\lim_{\delta\downarrow 0} \overline{\lim_{n\uparrow\infty}} \mathbb{P}(\tau_{w_n} < \delta w_n^2) = 0. \tag{43}$$

Due to (23), the scaling property of Brownian motion and since the local time is jointly continuous with probability 1, one gets for all $s \geq 0$ that

$$L(s,\cdot) \sim L(1, \cdot/\sqrt{s})\sqrt{s}. \tag{44}$$

Since

$$\psi(\delta w_n^2 \,|\, 0, \nu_n) = \frac{1}{2n} \sum_{k\in\mathbb{Z}} L\left(\delta w_n^2, \frac{S^{(n)}(k)}{n}\right),$$

by taking $s = \delta w_n^2$ in (44) one gets

$$\mathbb{P}\left(\psi(\delta w_n^2 \,|\, 0, \nu_n) \leq t\right) = \mathbb{P}(Y_n \leq t), \tag{45}$$

where

$$Y_n = \frac{1}{2n} \sum_{k\in\mathbb{Z}} L\left(1, \frac{S^{(n)}(k)}{\sqrt{\delta} S^{(n)}\left(\lfloor na \rfloor + \lfloor nx \rfloor\right)}\right) \frac{\sqrt{\delta} S^{(n)}\left(\lfloor na \rfloor + \lfloor nx \rfloor\right)}{n}.$$

Consider the event

$$\mathcal{B}_{\rho,\varepsilon} = \left\{L(1,y) \geq \varepsilon \;\forall y \in [0,\rho]\right\}.$$

On $\mathcal{B}_{\rho,\varepsilon}$ it holds

$$Y_n \geq \frac{\varepsilon\sqrt{\delta}}{2n} c_n(\sqrt{\delta}\rho) \frac{S^{(n)}\left(\lfloor na \rfloor + \lfloor nx \rfloor\right)}{n},$$

where

$$c_n(\sqrt{\delta}\rho) = \left|\left\{k \geq 0 \,:\, \frac{S^{(n)}\left(k + \lfloor nx \rfloor\right)}{S^{(n)}\left(\lfloor na \rfloor + \lfloor nx \rfloor\right)} \leq \sqrt{\delta}\rho\right\}\right|.$$

Due to condition (H) we can find a non negative sequence b_n such that

$$\lim_{n\uparrow\infty} \frac{b_n+1}{n}\,\frac{S^{(n)}\left(\lfloor an\rfloor+\lfloor xn\rfloor\right)}{n}=\infty, \tag{46}$$

$$\overline{\lim_{n\uparrow\infty}}\, \frac{S^{(n)}\left(b_n+\lfloor nx\rfloor\right)}{S^{(n)}\left(\lfloor na\rfloor+\lfloor nx\rfloor\right)}<\sqrt{\delta}\rho, \tag{47}$$

Due to (47), $c_n(\sqrt{\delta}\rho)\geq b_n+1$ for n large enough. Therefore, on $\mathcal{B}_{\rho,\varepsilon}$,

$$Y_n\geq \frac{\varepsilon\sqrt{\delta}(b_n+1)}{2n}\frac{S^{(n)}\left(\lfloor an\rfloor+\lfloor xn\rfloor\right)}{n}.$$

The above estimate and (46) imply that $\lim_{n\uparrow\infty} Y_n=\infty$ on $\mathcal{B}_{\rho,\varepsilon}$. In particular,

$$\overline{\lim_{n\uparrow\infty}}\, \mathbb{P}(Y_n\leq t)\leq \overline{\lim_{\rho\downarrow 0}}\,\overline{\lim_{\varepsilon\downarrow 0}}\,\mathbb{P}\left(\mathcal{B}^c_{\rho,\varepsilon}\right). \tag{48}$$

Since $L(1,0)>0$ and $L(1,\cdot)$ is continuous with probability 1, the r.h.s. in the above expression is zero. This result together with (42), (43) and (45) implies (39). Similarly one can prove that

$$\lim_{n\uparrow\infty}\mathbb{P}\left(X_n(t|x_n)<x_n-a\right)=0. \tag{49}$$

(39) and (49) imply that $X_n(t|x_n)\Rightarrow x$.

□

Remark 1. *Suppose that there exists an increasing sequence of positive integers* $\{n_k\}_{k\geq 1}$ *satisfying (12), (13) and (14). We claim that Proposition 6 remains true. To this aim observe that the arguments of the above proof together with (13) and (14) imply that*

$$\lim_{k\uparrow\infty}\sup_{t\leq T}\mathbb{P}\left(|X_{n_k}(t|x_{n_k})-x|>a\right)=0,\qquad \forall T>0, a>0. \tag{50}$$

For each integer n *set* $\underline{n}=n_K$, $\bar{n}=n_{K+1}$ *where* $K=\sup\{k\,:\,n_k\leq n\}$. *Due to the coupling discussed at the beginning of the above proof it holds*

$$X_{\underline{n}}(t(n/\underline{n})^2,x_{\underline{n}})\,\underline{n}/n\leq X_n(t,x_n)\leq X_{\bar{n}}(t(n/\bar{n})^2,x_{\bar{n}})\,\bar{n}/n.$$

In particular, for each $a>0$,

$$\mathbb{P}\left(X_n(t,x_n)>x+a\right)\leq \mathbb{P}\left(X_{\bar{n}}(t(n/\bar{n})^2,x_{\bar{n}})>(x+a)n/\bar{n}\right), \tag{51}$$

$$\mathbb{P}\left(X_n(t,x_n)<x-a\right)\leq \mathbb{P}\left(X_{\underline{n}}(t(n/\underline{n})^2,x_{\underline{n}})<(x-a)n/\underline{n}\right). \tag{52}$$

Note that due to (12) $\lim_{n\uparrow\infty} n/\underline{n}=\lim_{n\uparrow\infty} n/\bar{n}=1$. *Therefore, (50), (51) and (52) imply (37).*

5. Proof of Theorem 1 and Theorem 2

We point out that in both cases $\alpha\in(0,\infty)$ and $\alpha=\infty$ it holds

$$\frac{1}{n}\sum_{k\in\mathbb{Z}}\varphi\left(\frac{k}{n}\right)\sum_{j\in\mathbb{Z}}p(tn^2,k,j)\eta_j(0)=\frac{1}{n}\sum_{j\in\mathbb{Z}}\eta_j(0)\mathbb{E}\left(\varphi\left(X_n(t|j/n)\right)\right), \tag{53}$$

where $X_n(t|\cdot)$ has been defined in (29) for $\alpha\in(0,\infty)$ and in (36) for $\alpha=\infty$.

Let us first prove Theorem 1. Let $\varphi\in C_c(\mathbb{R})$ and set $g(x)=\mathbb{E}\left(\varphi\left(\sqrt{2/\alpha}B(t)+x\right)\right)$. Since $g\in C_0(\mathbb{R})\cap L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}}g(x)\rho_0(x)dx=\int_{\mathbb{R}}\varphi(x)\rho(x,t)dx,$$

due to our assumption on μ_n we get

$$\lim_{n\uparrow\infty}\mu_n\left(\left|\frac{1}{n}\sum_{k\in\mathbb{Z}}g\left(\frac{k}{n}\right)\eta_k-\int_{\mathbb{R}}\varphi(x)\rho(x,t)dx\right|>\delta\right)=0.$$

Due to the above limit, Proposition 2 and (53), in order to prove Theorem 1 it is enough to show that

$$\lim_{n\uparrow\infty}\mu_n\left(\frac{1}{n}\sum_{j\in\mathbb{Z}}\left|\mathbb{E}\left(\varphi\left(X_n(t|j/n)\right)\right)-\mathbb{E}\left(\varphi\left(\sqrt{2/\alpha}B(t)+j/n\right)\right)\right|\eta_j(0)\right)=0$$

Since the above expectation is bounded by

$$\frac{1}{n}\sum_{j\in\mathbb{Z}}\left|\mathbb{E}\left(\varphi\left(X_n(t|j/n)\right)\right)-\mathbb{E}\left(\varphi\left(\sqrt{2/\alpha}B(t)+j/n\right)\right)\right|,$$

due to Scheffé Theorem (see the arguments in [11] after Statement 15 or the proof of Theorem 2 below) and the uniform continuity of φ it is enough to prove that

$$\lim_{n\uparrow\infty}\mathbb{E}\left(\varphi\left(X_n(t|x_n)\right)\right)=\mathbb{E}\left(\varphi\left(\sqrt{2/\alpha}B(t)+x\right)\right),\qquad\forall x\in\mathbb{R},\tag{54}$$

where $x_n=\lfloor xn\rfloor$. The above limit follows from Proposition 5, thus concluding the proof of Theorem 1.

Let us prove Theorem 2. Due to (10), Proposition 2 and (53), it is enough to prove that

$$\lim_{n\uparrow\infty}\mu_n\left(\left|\frac{1}{n}\sum_{j\in\mathbb{Z}}\mathbb{E}\left(\varphi\left(X_n(t|j/n)\right)\right)\eta_j-\frac{1}{n}\sum_{j\in\mathbb{Z}}\varphi\left(\frac{j}{n}\right)\eta_j\right|\right)=0.$$

Without loss of generality we can assume $\varphi\geq 0$. Then the above expectation is bounded by

$$\int_{\mathbb{R}}\left|f_n(x)-\varphi(x)\right|dx+\varepsilon_n\tag{55}$$

where $\lim_{n\uparrow\infty}\varepsilon_n=0$ and

$$f_n(x)=\sum_{j\in\mathbb{Z}}\mathbb{E}\left(\varphi\left(X_n(t|j/n)\right)\right)\mathbb{I}_{\{x\in[j/n,(j+1)/n)\}}.$$

In order to conclude it is enough to apply the same arguments described in [11] after Statement 15: $f_n\geq 0$, $\lim_{n\uparrow\infty}f_n(x)=\varphi(x)$ for all $x\in\mathbb{R}$ due to Proposition 6 and

$$\begin{aligned}\int_{\mathbb{R}}f_n(x)dx=\frac{1}{n}\sum_{j\in\mathbb{Z}}\mathbb{E}\left(\varphi\left(X_n(t|j/n)\right)\right)&=\frac{1}{n}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}p(tn^2,j,k)\varphi\left(\frac{k}{n}\right)\\&=\frac{1}{n}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}p(tn^2,k,j)\varphi\left(\frac{k}{n}\right)=\frac{1}{n}\sum_{k\in\mathbb{Z}}\varphi\left(\frac{k}{n}\right)\to\int_{\mathbb{R}}\varphi(x)dx.\end{aligned}$$

In particular we can apply Scheffé Theorem and get that the integral in (55) goes to 0, thus allowing to conclude the proof of Theorem 2.

6. Proof of Proposition 1

Since c_k is in the domain of attraction of a ν–stable law we can write (see [4][Theorem 2.2.8])

$$\mathbb{P}\left(\frac{1}{c_k} > y\right) = L(y) y^{-\nu}, \qquad \forall y \geq 1,$$

with L slowly varying function. In particular (see [4][Theorem A3.3]) L can be written as

$$L(x) = h(x) \exp\left\{\int_1^x \frac{g(u)}{u} du\right\}, \qquad \forall x \geq 1, \tag{56}$$

for suitable functions h, g such that $h > 0$, $\lim_{x\uparrow\infty} h(x) = h_0 \in (0, \infty)$, $\lim_{x\uparrow\infty} g(x) = 0$.

The proof of Proposition 1 is based on the following lemma:

Lemma 4. *Let $Y_1, Y_2, \dots$ be a sequence of i.i.d. random variables distributed as $1/c_0$ and set*

$$M_n = \max\{Y_1, Y_2, \dots, Y_n\}, \qquad S_n = \sum_{k=1}^n Y_k.$$

Then, given $\beta, \delta, \gamma > 0$, there exist positive constants c_1, c_2 such that for all $n \geq 1$

$$\mathbb{P}\left(M_n \leq n^{\frac{1}{\nu}-\delta}\right) \leq \exp\left\{-c_1 n^{\delta\nu}\right\}, \tag{57}$$

$$\mathbb{P}\left(S_{\lfloor n^\beta \rfloor} \geq n^\gamma\right) \leq c_2 n^{\beta-\nu\gamma+\delta}. \tag{58}$$

Proof. Since $1 - z \leq e^{-z}$ for all $z \geq 0$ and due to (56)

$$\mathbb{P}\left(M_n \leq n^{\frac{1}{\nu}-\delta}\right) \leq \left(1 - L\left(n^{\frac{1}{\nu}-\delta}\right) n^{\delta\nu-1}\right)^n \leq \exp\left\{-L\left(n^{\frac{1}{\nu}-\delta}\right) n^{\delta\nu}\right\}.$$

Therefore (57) holds with $c_1 = \inf_{x\geq 1} L(x)$ which is positive due to representation (56).

In order to prove (58) we point out that, due to (56),

$$\lim_{x\uparrow\infty} L(x)/x^u = 0, \qquad \forall u > 0.$$

Due to the above observation and since $1 - z \geq e^{-2z}$ for z small enough, we get for $n \geq n_0$

$$\begin{aligned}\mathbb{P}\left(M_{\lfloor n^\beta \rfloor} \geq n^\gamma\right) = 1 - \left(1 - L(n^\gamma) n^{-\nu\gamma}\right)^{\lfloor n^\beta \rfloor} &\leq 1 - \exp\left\{-2L(n^\gamma) \lfloor n^\beta \rfloor n^{-\nu\gamma}\right\} \\ &\leq 2L(n^\gamma) \lfloor n^\beta \rfloor n^{-\nu\gamma} \leq n^{\beta-\nu\gamma+\delta}.\end{aligned} \tag{59}$$

Moreover, by integration by parts,

$$\mathbb{E}\left(Y_1 \mathbb{I}_{Y_1 \leq n^\gamma}\right) \leq 1 + \int_1^{n^\gamma} x d\left(L(x) x^{-\nu}\right) \leq 1 + L(n^\gamma) n^{\gamma(1-\nu)} \leq c_3 n^{\gamma(1-\nu)+\delta}. \tag{60}$$

In particular

$$\mathbb{P}\left(S_{\lfloor n^\beta \rfloor} \geq n^\gamma,\ M_{\lfloor n^\beta \rfloor} \leq n^\gamma\right) \leq \mathbb{P}\left(\sum_{j=1}^{\lfloor n^\beta \rfloor} Y_j \mathbb{I}_{Y_j \leq n^\gamma} \geq n^\gamma\right) \leq n^{\beta-\gamma} \mathbb{E}\left(Y_1 \mathbb{I}_{Y_1 \leq n^\gamma}\right) \leq c_3 n^{\beta-\nu\gamma+\delta}. \tag{61}$$

Bounds (59) and (61) imply that

$$\mathbb{P}\left(S_{\lfloor n^\beta \rfloor} \geq n^\gamma\right) \leq \mathbb{P}\left(M_{\lfloor n^\beta \rfloor} \geq n^\gamma\right) + \mathbb{P}\left(S_{\lfloor n^\beta \rfloor} \geq n^\gamma,\ M_{\lfloor n^\beta \rfloor} \leq n^\gamma\right) \leq c_2 n^{\beta-\nu\gamma+\delta}.$$

□

In order to prove Proposition 1 we distinguish between the cases $0 < \nu < 1/2$ and $1/2 \le \nu < 1$.

If $0 < \nu < 1/2$, then take $\delta > 0$ such that $2 + \delta < 1/\nu$. Due to (57) and Borel–Cantelli lemma for almost all environments, given $x, a \in \mathbb{Q}$ with $a \neq 0$, there exists a positive constant c such that

$$S\left(\lfloor an \rfloor + \lfloor xn \rfloor\right) - S\left(\lfloor xn \rfloor\right) > cn^{1/\nu - \delta}, \qquad \forall n \ge 1.$$

The above estimate implies conditions (7) and (8) with $b_n = 0$. Hence it is enough to apply Theorem 2.

If $1/2 \le \nu < 1$, set

$$\begin{cases} \beta = 2 - 1/\nu + 2\delta, \\ \gamma = 1/\nu - 2\delta/\nu, \end{cases}$$

with $\delta > 0$ small enough to have $5\delta < 1/\nu - 1$ and $2\delta < 1$. Then $\beta, \gamma > 0$ and

$$\beta - \nu\gamma + \delta = 1 - 1/\nu + 5\delta < 0.$$

Set $n_k = \lfloor k^\rho \rfloor$ and $b_k = \lfloor n_k^\beta \rfloor$, where $\rho > 0$ is large enough to have $\rho(1 - 1/\nu + 5\delta) < -1$. Due to this choice and by (57), (58) and Borel–Cantelli lemma for almost all environments the following holds:

Given $x, a \in \mathbb{Q}$ with $a \neq 0$, there exist positive constants c_1, c_2 such that for all $k \ge 1$

$$S\left(\lfloor an_k \rfloor + \lfloor xn_k \rfloor\right) - S\left(\lfloor xn_k \rfloor\right) > c_1 n_k^{1/\nu - \delta}, \tag{62}$$

$$S\left(b_k + \lfloor xn_k \rfloor\right) - S\left(\lfloor xn_k \rfloor\right) < c_2 n_k^{\gamma}. \tag{63}$$

In particular, the l.h.s. of (13) with fixed k is bounded from below by $c_1 \lfloor n_k^\beta \rfloor n^{1/\nu - 2 - \delta}$ while the l.h.s. of (14) with fixed k is bounded from above by $(c_2/c_1) n_k^{\gamma - 1/\nu + \delta}$. Due to our choice of β, γ conditions (13) and (14) are satisfied. Hence the thesis follows from the discussion after Theorem 2

Appendix A. A counterexample

As already noted, (2) implies condition (4). We show here that the inverse implication is false if $\alpha = \infty$.

Consider the subsets $A, B, C \subset \mathbb{Z}$ defined as

$$A = \cup_{n=0}^{\infty} \left[2^{2n}, 2^{2n+1}\right) \cap \mathbb{Z}, \qquad B = A \cup (-A), \qquad C = \mathbb{Z} \setminus B.$$

Define

$$c_j = \begin{cases} 1 & \text{if } j \in B, \\ e^{-|j|} & \text{if } j \notin B. \end{cases}$$

Then

$$\lim_{n \uparrow \infty} \frac{1}{2^{2n} - 1} \sum_{j=2^{2n}}^{2^{2n+1}-2} \frac{1}{c_j} = 1.$$

In particular, (2) cannot hold for $\alpha = \infty$, $x = 1$, $y = 2$. Let us verify that (4) is fulfilled for $\alpha = \infty$. Given $k > 1$,

$$\sum_{j=1}^{k-1} \frac{1}{c_j} \ge \sum_{j=1}^{N_k} e^j = \frac{e^{N_k + 1} - e}{e - 1} \tag{64}$$

where $N_k = |[1,k) \cap B^c|$. We claim that

$$\inf_{k>1} \frac{N_k}{k} > 0. \tag{65}$$

In fact, if $2^{2n} \leq k < 2^{2n+1}$ then

$$[1,k) \cap B^c = \cup_{u=1}^n [2^{2u-1}, 2^{2u}) \cap \mathbb{Z},$$

which implies that

$$N_k = \sum_{u=1}^n 2^{2u-1} = \frac{2}{3}(2^{2n} - 1). \tag{66}$$

For $2^{2n+1} \leq k < 2^{2n+2}$ the above identity implies that

$$\frac{2}{3}(2^{2n} - 1) = N_{2^{2n}} \leq N_k \leq N_{2^{2n+2}} = \frac{2}{3}(2^{2n+2} - 1). \tag{67}$$

(65) follows from (66) and (67).

Due to (64) and (65) we get

$$\liminf_{k\uparrow\infty} \frac{1}{k} \sum_{j=1}^{k-1} \frac{1}{c_j} \geq \left(\inf_{n>1} \frac{N_n}{n} \right) \liminf_{N\uparrow\infty} \frac{e^{N+1} - e}{N(e-1)} = \infty.$$

By symmetry one gets

$$\lim_{k\uparrow\infty} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{c_{-j}} = \infty.$$

In conclusion, $\{c_k : k \in \mathbb{Z}\}$ satisfies (4) with $\alpha = \infty$ but it does not satisfy (2) with $\alpha = \infty$.

Appendix B. Exclusion processes with bond disorder on $\mathbb{Z}^d$

As already noted in the Introduction, the results described in Section 3 can be easily generalized to higher dimension and by the arguments of Section 5 one can prove the hydrodynamic limit of the exclusion process with bond disorder on $\mathbb{Z}^d$ when having suitable information on the associated random walk. In order to clarify this point, we recall in this Appendix the main steps of our method leading to the proof of Theorem 1, which can be easily extended to higher dimension.

To this aim we denote by $\mathcal{E}$ the set of non oriented bonds of $\mathbb{Z}^d$ and consider the exclusion process on $\mathbb{Z}^d$ with generator

$$\mathcal{L}f(\eta) = \sum_{b\in\mathcal{E}} c_b \left(f(\eta^b) - f(\eta) \right), \tag{68}$$

where $c_b \geq 0$ and η^b is the configuration obtained from η by switching the values at the vertexes of b. Due to a standard percolation argument applied to the graphical representation of exclusion processes, if $\sup_{b\in\mathcal{E}} c_b < \infty$ then the above exclusion process is well defined.

It is simple to check that all the arguments and the results of Section 3 remain valid in higher dimension when suitably changing the notation. In particular, the analogous of

Proposition 2 holds: Given $\delta > 0$, $t > 0$, $\varphi \in C_c(\mathbb{R}^d)$ and given a sequence of probability measures μ_n on $\{0,1\}^{\mathbb{Z}^d}$,

$$\lim_{n\uparrow\infty} \mathbb{P}_{\mu_n}\left(\left|\frac{1}{n^d}\sum_{k\in\mathbb{Z}^d}\varphi\left(\frac{k}{n}\right)\eta_k(tn^2) - \frac{1}{n^d}\sum_{j\in\mathbb{Z}^d}\eta_j(0)P_t^n\varphi\left(\frac{j}{n}\right)\right| > \delta\right) = 0, \tag{69}$$

where $P_t^n\varphi(j/n) = \mathbb{E}\left(\varphi\left(X_n(t|j/n)\right)\right)$ and $X_n(t|j/n)$ is the random walk on $\mathbb{Z}^d/n$ starting at j/n with generator H_n defined as the d–dimensional version of (30), namely

$$H_n f\left(j/n\right) = n^2 \sum_{k\in\mathbb{Z}^d:\, \|k-j\|_\infty = 1} c_{\{k,j\}}\left(f\left(k/n\right) - f\left(j/n\right)\right), \qquad \forall j \in \mathbb{Z}^d. \tag{70}$$

Suppose that

$$\lim_{n\uparrow\infty} \mu_n\left(\frac{1}{n^d}\left|\sum_{j\in\mathbb{Z}^d}\eta_j(0)P_t^n\varphi\left(\frac{j}{n}\right) - \sum_{j\in\mathbb{Z}^d}\eta_j(0)P_t\varphi\left(\frac{j}{n}\right)\right| \geq \delta\right) = 0, \qquad \forall \varphi \in C_c(\mathbb{R}^d), \tag{71}$$

where $P_t\varphi(x) = \mathbb{E}\left(\varphi(x + W(t))\right)$ and W is a Brownian motion on $\mathbb{R}^d$ starting at 0 (one could even consider other limiting processes). Trivially, (71) holds if

$$\lim_{n\uparrow\infty}\frac{1}{n^d}\sum_{j\in\mathbb{Z}^d}\left|P_t^n\varphi(j/n) - P_t\varphi(j/n)\right| = 0, \qquad \forall\varphi\in C_c(\mathbb{R}^d). \tag{72}$$

Note that in Section 5 we have derived (72) from Proposition 5 and Scheffé Theorem (the same method works in all dimensions).

At this point, when having (71) or even (72), the d–dimensional version of Theorem 1 is easily proven and the hydrodynamic equation coincides with the linear heat equation associated to W.

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Note added in proof. After completing this work, we were aware of [7] where Theorem 1 is proven by a different approach.

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